

Extension of Chern-Simons Forms

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Abstract

We investigate metric independent, gauge invariant and closed forms in the generalized YM theory. These forms are polynomial on the corresponding fields strength tensors - curvature forms and are analogous to the Pontryagin-Chern densities in the YM gauge theory. The corresponding secondary characteristic classes have been expressed in integral form in analogy with the Chern-Simons form. Because they are not unique, the secondary forms can be dramatically simplified by the addition of properly chosen differentials of one-step-lower-order forms. Their gauge variation can also be found yielding the potential anomalies in the gauge field theory.

1 Introduction

The chiral anomalies, Abelian and non-Abelian [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 18], can be derived by a differential geometric method without having to evaluate Feynman diagrams. Indeed, the non-Abelian anomaly in $(2n - 2)$ -dimensional space-time may be obtained from the Abelian anomaly in $2n$ dimensions by a series of reduction (transgression) steps [6, 7, 8, 9, 10, 11, 12, 13, 18]. The $U_A(1)$ gauge anomaly is given by the Pontryagin-Chern-Simons $2n$ -form [6, 7, 8, 9, 10, 11, 12, 13, 18]:

$$d * J^A \propto \mathcal{P}_{2n} = \text{Tr}(G^n) = d \omega_{2n-1}, \quad (1.1)$$

where ω_{2n-1} is the Chern-Simons form in $2n - 1$ dimensions [6, 7, 12]:

$$\omega_{2n-1}(A) = n \int_0^1 dt \text{Str}(A, G_t^{n-1}), \quad (1.2)$$

$G = dA + A^2$ is the 2-form Yang-Mills (YM) field-strength tensor of the 1-form vector field¹ $A = -igA_\mu^a L^a dx^\mu$ and $G_t = tG + (t^2 - t)A^2$. The non-Abelian anomaly [1, 2, 3, 4, 5] can be obtained by the gauge variation of ω_{2n-1} [6, 7, 8, 9, 10, 12, 13, 16, 17, 18]:

$$\delta \omega_{2n-1} = d\omega_{2n-2}^1, \quad (1.3)$$

where the $(2n - 2)$ -form has the following integral representation [6, 7]:

$$\omega_{2n-2}^1(\xi, A) = n(n-1) \int_0^1 dt(1-t) \text{Str}(d\xi, A, G_t^{n-2}). \quad (1.4)$$

Here $\xi = \xi^a L_a$ is a scalar gauge parameter and Str denotes a symmetrized trace². The covariant divergence of the non-Abelian left and right handed currents is given by this $(2n - 2)$ -form.

In recent articles [19, 20, 21] the authors found closed invariant forms similar to the Pontryagin-Chern-Simons forms in non-Abelian tensor gauge field theory [22, 23, 24]. The first series of closed invariant forms are defined in $\mathcal{D} = 2n + 4$ dimensions and are given by the expression

$$\Phi_{2n+4} = \text{tr}(G_4 G^n) = \text{Str}(G_4, G^n) = d\psi_{2n+3}, \quad (1.6)$$

where the corresponding secondary $(2n + 3)$ -form ψ_{2n+3} is in $\mathcal{D} = 2n + 3$ dimensions

$$\psi_{2n+3} = \text{Str}(A_3, G^n) \quad (1.7)$$

¹ L^a are the generators of the Lie algebra.

²In this article we shall use the symmetrized trace

$$\text{Str}(A_1, A_2, \dots, A_n) \equiv \frac{1}{n!} \sum_{(i_1, \dots, i_n)} (A_{i_1} A_{i_2} \dots A_{i_n}), \quad (1.5)$$

where the sum is over all permutations. Its properties are described in the Appendix B of the article [10].

and $G_4 = dA_3 + \{A, A_3\}^3$. It turns out that the introduction of Str in the above equations leads to very crucial simplifications in all our subsequent derivations. For compact notation, when some of the entries of Str are the same, we write them in power form. The second series of forms is defined in $\mathcal{D} = 2n + 6$ dimensions [21]:

$$\Xi_{2n+6} = Str(G_6, G^n) + nStr(G_4^2, G^{n-1}) = d\phi_{2n+5}. \quad (1.8)$$

The general expression for the secondary $(2n + 5)$ -form ϕ_{2n+5} will be constructed in this article. The third series of invariant closed forms found in this article Υ_{2n+8} in $\mathcal{D} = 2n + 8$ dimensions is

$$\Upsilon_{2n+8} = Str(G_8, G^n) + 3nStr(G_4, G_6, G^{n-1}) + n(n-1)Str(G_4^3, G^{n-2}) = d\rho_{2n+7}. \quad (1.9)$$

Its secondary form ρ_{2n+7} will be presented in the next sections.

All forms Φ_{2n+4} , Ξ_{2n+6} and Υ_{2n+8} are analogous to the Pontryagin-Chern-Simons densities \mathcal{P}_{2n} in the YM gauge theory (1.1) in the sense that they are *gauge invariant, closed and metric independent*. Our aim is to investigate this rich class of topological invariants of extended gauge theory as well as to find out potential gauge anomalies performing transgressions analogous to (1.1) and (1.3):

$$\mathcal{P}_{2n} \Rightarrow \omega_{2n-1} \Rightarrow \omega_{2n-2}^1. \quad (1.10)$$

Therefore we shall perform the following transgressions:

$$\begin{aligned} \Phi_{2n+4} &\Rightarrow \psi_{2n+3} \Rightarrow \psi_{2n+2}^1, \\ \Xi_{2n+6} &\Rightarrow \phi_{2n+5} \Rightarrow \phi_{2n+4}^1, \\ \Upsilon_{2n+8} &\Rightarrow \rho_{2n+7} \Rightarrow \rho_{2n+6}^1. \end{aligned} \quad (1.11)$$

We shall find explicit expressions for these primary invariants in terms of higher order polynomials of the curvature forms on a vector bundle. The most difficult challenge will be the evaluation and differentiation of the very complicated noncommutative polynomial expressions as well as the search of the most simple expressions for the secondary forms. The secondary forms are not uniquely defined. Indeed, the secondary form ψ_{2n+3} is defined modulo the exterior derivative of an arbitrary $(2n + 2)$ -form $\psi_{2n+3} \sim \psi_{2n+3} + d\alpha_{2n+2}$, the form ϕ_{2n+5} modulo the exterior derivative of a $(2n + 4)$ -form $\phi_{2n+5} \sim \phi_{2n+5} + d\beta_{2n+4}$ and the form ρ_{2n+7} modulo the exterior derivative of a $(2n + 6)$ -form $\rho_{2n+7} \sim \rho_{2n+7} + d\gamma_{2n+6}$. When the difference of two closed forms is an exact form, they are said to be cohomologous to each other. Therefore the problem is to find out the most simple representatives in the set of equivalence classes. Conveniently chosen exact forms will dramatically simplify the

³In the Appendix one can find the definition of tensor gauge fields and the corresponding curvature forms.

expressions. These problems will be solved by using properties of symmetrized traces (1.5) defined in [10].

In Section 2 we shall present a general construction and analysis of the primary forms Φ_{2n+4} and Ξ_{2n+6} , their secondary forms ψ_{2n+3} and ϕ_{2n+5} and the corresponding anomalies represented by ψ_{2n+2}^1 and ϕ_{2n+4}^1 . The material of this section is not completely new, but the alternative derivation in terms of symmetrized traces will allow to extend the results to the higher-dimensional forms Υ_{2n+8} . In Section 3 we shall derive the explicit expressions for the primary form Υ_{2n+8} , written in terms of the symmetrized traces (1.5), starting from the low-dimensional forms listed in [21]. Next, we shall find the secondary forms ρ_{2n+7} and the corresponding gauge anomalies ρ_{2n+6}^1 associated with each of the independent gauge transformations $\delta_\xi, \delta_{\zeta_2}, \delta_{\zeta_4}, \delta_{\zeta_6}$. In the conclusion we summarize the primary and secondary invariant forms constructed in the article. In the Appendix we present useful formulas for the gauge transformations of the fields, the corresponding Bianchi identities and a one-parameter deformation of fields generalizing deformation of [6, 7].

2 Gauge and Metric Independent Forms

We shall start by deriving the already known results for the form Φ_{2n+4} using the properties of the symmetrized traces. This approach will allow to extend the derivation to more complicated cases. Indeed, the form can be represented in terms of a symmetrized trace as [20, 21]

$$\Phi_{2n+4} = Str(G_4, G^n) \quad (2.1)$$

so that its gauge invariance with respect to the standard-scalar gauge transformations δ_ξ and the tensor gauge transformations δ_{ζ_2} can be easily checked:

$$\begin{aligned} \delta_\xi \Phi_{2n+4} &= Str(\delta_\xi G_4, G^n) + n Str(G_4, \delta_\xi G, G^{n-1}) = \\ &= Str([G_4, \xi], G^n) + n Str(G_4, [G, \xi], G^{n-1}) = 0, \\ \delta_{\zeta_2} \Phi_{2n+4} &= Str(\delta_{\zeta_2} G_4, G^n) = Str([G, \zeta_2], G^n) = \\ &= \frac{1}{n+1} \left[Str([G, \zeta_2], G^n) + \dots + Str(G^n, [G, \zeta_2]) \right] = 0. \end{aligned} \quad (2.2)$$

On the last steps we used the identity (B.10) of the Appendix B of the article [10]. Our next step is to check that Φ_{2n+4} is a closed form. Indeed,

$$\begin{aligned} d\Phi_{2n+4}(A, A_3) &= Str(DG_4, G^n) + n Str(G_4, DG, G^{n-1}) = Str([G, A_3], G^n) = \\ &= \frac{1}{n+1} \left[Str([G, A_3], G^n) + Str(G, [G, A_3], G^{n-1}) + \dots + Str(G^n, [G, A_3]) \right] = 0, \end{aligned}$$

where we used the Bianchi identity $DG = 0$. To find out the secondary form we shall use a one-parameter deformation of the gauge potentials: [6, 20, 21]

$$A_t = tA, \quad A_{3t} = tA_3, \quad A_{5t} = tA_5, \quad A_{7t} = tA_7, \quad A_{9t} = tA_9, \dots,$$

defined in the Appendix (5.6), (5.7), take the derivative and employ (B.13) of [10]:

$$\begin{aligned} \frac{d}{dt} \Phi_{2n+4}(A_t, A_{3t}) &= Str\left(\frac{dG_{4t}}{dt}, G_t^n\right) + nStr\left(G_{4t}, \frac{dG_t}{dt}, G_t^{n-1}\right) = \\ &= Str\left(D_t A_3 + t\{A, A_3\}, G_t^n\right) + nStr\left(G_{4t}, D_t A, G_t^{n-1}\right) = \\ &= dStr\left(A_3, G_t^n\right) + ndStr\left(G_{4t}, A, G_t^{n-1}\right) + \\ &\quad + Str\left(\{A, A_{3t}\}, G_t^n\right) - nStr\left([G_t, A_{3t}], A, G_t^{n-1}\right) = \\ &= d\left\{Str\left(A_3, G_t^n\right) + nStr\left(G_{4t}, A, G_t^{n-1}\right)\right\}. \end{aligned} \quad (2.3)$$

Integrating the above equation over the parameter t in the interval $[0, 1]$ we get the integral representation of the secondary form

$$\psi_{2n+3} = \int_0^1 dt \left[Str\left(A_3, G_t^n\right) + nStr\left(G_{4t}, A, G_t^{n-1}\right) \right]. \quad (2.4)$$

This secondary form is not unique. It can be modified by the addition of the differential of a one-step-lower-order form $d\alpha_{2n+2}$. Two closed forms which differ by an exact form are said to be cohomologous to each other. The integral on the right hand side of the equation looks complicated, but if we add a properly chosen exact form $d\alpha_{2n+2}$, then it will be dramatically simplified. Let us take it in the following form:

$$\alpha_{2n+2} = -n \int_0^1 dt Str\left(A_{3t}, A, G_t^{n-1}\right).$$

Then we get:

$$\begin{aligned} \psi_{2n+3} &\sim \psi_{2n+3} + d\alpha_{2n+2} = \psi_{2n+3} - n \int_0^1 dt Str\left(D_t A_{3t}, A, G_t^{n-1}\right) + \\ &\quad + n \int_0^1 dt Str\left(A_{3t}, D_t A, G_t^{n-1}\right) - n(n-1) \int_0^1 dt Str\left(A_{3t}, A, D_t G_t, G_t^{n-2}\right), \end{aligned}$$

where $D_t G_t = 0$ ⁴. Using the relations (5.2), (5.7) and the integral representation (2.4), we have:

$$\begin{aligned} \psi_{2n+3} &\sim \psi_{2n+3} - n \int_0^1 dt Str\left(G_{4t}, A, G_t^{n-1}\right) + n \int_0^1 dt Str\left(A_{3t}, \frac{\partial G_t}{\partial t}, G_t^{n-1}\right) = \\ &= \int_0^1 dt \left\{ Str\left(A_3, G_t^n\right) + nStr\left(A_{3t}, \frac{\partial G_t}{\partial t}, G_t^{n-1}\right) \right\} = \int_0^1 dt \frac{\partial}{\partial t} Str\left(A_{3t}, G_t^n\right) = Str\left(A_3, G^n\right). \end{aligned} \quad (2.5)$$

⁴The symbol " \sim " denotes the cohomology relation between the two forms.

Thus, the secondary form gets the following compact form

$$\psi_{2n+3} = Str(A_3, G^m). \quad (2.6)$$

The secondary forms (2.4) and (2.6) are representatives of the same cohomology class, because their difference is an exact form $d\alpha_{2n+2}$, but, as one can see (2.6), has a much more simple expression. Using (5.2), (5.5) we can verify that $d\psi_{2n+3} = \Phi_{2n+4}$:

$$dStr(A_3, G^m) = Str(DA_3, G^m) - nStr(A_3, DG, G^{m-1}) = Str(G_4, G^m).$$

The form (2.6) allows to find the potential anomalies of the theory, by the following transgression steps. The gauge invariance of the primary form Φ_{2n+4} means that $\delta\Phi_{2n+4} = d(\delta\psi_{2n+3}) = 0$. By employing the Poincare's lemma it follows that $\delta\psi_{2n+3} = d\psi_{2n+2}^1$, where ψ_{2n+2}^1 is the potential anomaly. Thus, in order to proceed we have to calculate the gauge variation of the secondary form with respect to the gauge transformations δ_ξ and δ_{ζ_2} . We have⁵

$$\begin{aligned} \delta_\xi \psi_{2n+3} &= Str(\delta_\xi A_3, G^m) + nStr(A_3, \delta_\xi G, G^{m-1}) = \\ &= Str([A_3, \xi], G^m) + nStr(A_3, [G, \xi], G^{m-1}) = 0, \end{aligned} \quad (2.7)$$

that is, the secondary form is gauge invariant with respect to standard-scalar gauge transformations δ_ξ and therefore there are no gauge anomalies associated with the scalar gauge transformations. But with respect to the tensor gauge transformations δ_{ζ_2} there are anomalies

$$\delta_{\zeta_2} \psi_{2n+3} = Str(\delta_{\zeta_2} A_3, G^{m-1}) = Str(D\zeta_2, G^m) = dStr(\zeta_2, G^m). \quad (2.8)$$

Therefore the anomaly is

$$\psi_{2n+2}^{(1)}(\zeta_2, A) = Str(\zeta_2, G^m). \quad (2.9)$$

In summary, we have the expressions (2.1) for primary form Φ_{2n+4} , the expression (2.6) for the secondary form ψ_{2n+3} and (2.9) for the anomaly.

We shall now move to the next primary form Ξ_{2n+6} , which can be written in terms of symmetrized traces as [21]:

$$\Xi_{2n+6} = Str(G_6, G^m) + nStr(G_4^2, G^{m-1}). \quad (2.10)$$

Each term of this expression is independently gauge invariant. Indeed,

$$\begin{aligned} \delta_\xi Str(G_6, G^m) &= Str(\delta_\xi G_6, G^m) + nStr(G_6, \delta_\xi G, G^{m-1}) = \\ &= Str([G_6, \xi], G^m) + nStr(G_6, [G, \xi], G^{m-1}) = 0, \end{aligned} \quad (2.11)$$

⁵The identity (B.10) of the Appendix B [10] should be used.

and for the second term we shall get

$$\begin{aligned}\delta_\xi Str\left(G_4^2, G^{n-1}\right) &= 2Str\left(\delta_\xi G_4, G_4, G^{n-1}\right) + (n-1)Str\left(G_4^2, \delta_\xi G, G^{n-2}\right) \\ &= 2Str\left([G_4, \xi], G_4, G^{n-1}\right) + (n-1)Str\left(G_4^2, [G, \xi], G^{n-2}\right) = 0,\end{aligned}$$

where in the last two equations we again used (B.10) of [10]. However only the sum of these terms is a closed form. We can check the closeness of the form Ξ_{2n+6} by taking the exterior derivative:

$$\begin{aligned}d\Xi_{2n+6} &= Str\left(DG_6, G^n\right) + nStr\left(G_6, DG, G^{n-1}\right) + \\ &+ 2nStr\left(DG_4, G_4, G^{n-1}\right) + n(n-1)Str\left(G_4^2, DG, G^{n-2}\right) = \\ &= 2Str\left([G_4, A_3], G^n\right) + Str\left([G, A_5], G^n\right) + 2nStr\left([G, A_3], G_4, G^{n-1}\right) = \quad (2.12) \\ &= 2\left\{Str\left([G_4, A_3], G^n\right) + nStr\left(G_4, [G, A_3], G^{n-1}\right)\right\} + Str\left([G, A_5], G^n\right) = 0.\end{aligned}$$

On the first step we used (B.13) of [10] and on the second - the Bianchi identities $DG = 0$. On the last step, the terms in the big brace as well as the last term are zero because of (B.10) of [10]. Again, according to Poincaré's lemma, this equation implies that Ξ_{2n+6} can be locally written as an exterior derivative of a certain $(2n+5)$ -form. In order to find that form we need to differentiate Ξ_{2n+6} over the deformation parameter t . We have:

$$\begin{aligned}\frac{d}{dt}\Xi_{2n+6}(A_t, A_{3t}, A_{5t}) &= Str\left(\frac{dG_{6t}}{dt}, G_t^n\right) + nStr\left(G_{6t}, \frac{dG_t}{dt}, G_t^{n-1}\right) + 2nStr\left(\frac{dG_{4t}}{dt}, G_{4t}, G_t^{n-1}\right) + \\ &+ n(n-1)Str\left(G_{4t}^2, \frac{dG_t}{dt}, G_t^{n-2}\right) = Str\left(D_t A_5, G_t^n\right) + Str\left(\{A, A_{5t}\}, G_t^n\right) + 2Str\left(\{A_3, A_{3t}\}, G_t^n\right) + \\ &+ nStr\left(G_{6t}, D_t A, G_t^{n-1}\right) + 2nStr\left(D_t A_3, G_{4t}, G_t^{n-1}\right) + 2nStr\left(\{A, A_{3t}\}, G_{4t}, G_t^{n-1}\right) + \\ &+ n(n-1)Str\left(G_{4t}^2, D_t A, G_t^{n-2}\right) = dStr\left(A_5, G_t^n\right) + Str\left(\{A, A_{5t}\}, G_t^n\right) + 2Str\left(\{A, A_{3t}\}, G_t^n\right) + \\ &+ ndStr\left(G_{6t}, A, G_t^{n-1}\right) - nStr\left(D_t G_{6t}, A, G_t^{n-1}\right) + \\ &+ 2ndStr\left(A_3, G_{4t}, G_t^{n-1}\right) + 2nStr\left(A_3, D_t G_{4t}, G_t^{n-1}\right) + 2nStr\left(\{A, A_{3t}\}, G_{4t}, G_t^{n-1}\right) \\ &+ n(n-1)dStr\left(G_{4t}^2, A, G_t^{n-2}\right) - 2n(n-1)Str\left(D_t G_{4t}, G_{4t}, A, G_t^{n-2}\right) =\end{aligned}$$

$$\begin{aligned}
&= d \left\{ n \text{Str} \left(G_{6t}, A, G_t^{n-1} \right) + n(n-1) \text{Str} \left(G_{4t}^2, A, G_t^{n-2} \right) + 2n \text{Str} \left(G_{4t}, A_3, G_t^{n-1} \right) + \text{Str} \left(A_5, G_t^n \right) \right\} + \\
&\quad + 2 \left[\text{Str} \left(\{A_3, A_{3t}\}, G_t^n \right) + n \text{Str} \left(A_3, [G_t, A_{3t}], G_t^{n-1} \right) + \right. \\
&\quad \left. + 2n \left[\text{Str} \left(\{A, A_{3t}\}, G_{4t}, G_t^{n-1} \right) - \text{Str} \left([G_{4t}, A_{3t}], A, G_t^{n-1} \right) + \right. \right. \\
&\quad \left. \left. + 2n \left[\text{Str} \left(\{A, A_{3t}\}, G_{4t}, G_t^{n-1} \right) + \text{Str} \left(A, [G_{4t}, A_{3t}], G_t^{n-1} \right) - (n-1) \text{Str} \left([G_t, A_{3t}], G_{4t}, A, G_t^{n-2} \right) \right] + \right. \right. \\
&\quad \left. \left. + \text{Str} \left(\{A, A_{5t}\}, G_t^n \right) - n \text{Str} \left([G_t, A_{5t}], A, G_t^{n-1} \right) \right] = d\phi_{2n+5}. \tag{2.13}
\end{aligned}$$

On the second, third and last steps we used the relations (5.7) and (B.13), (B.10) of [10] respectively. Integrating the above equation over the parameter t in the interval $[0, 1]$ we shall get the following integral representation of the secondary form:

$$\begin{aligned}
\phi_{2n+5} &= \int_0^1 dt \left\{ n \text{Str} \left(G_{6t}, A, G_t^{n-1} \right) + n(n-1) \text{Str} \left(G_{4t}^2, A, G_t^{n-2} \right) + \right. \\
&\quad \left. + 2n \text{Str} \left(G_{4t}, A_3, G_t^{n-1} \right) + \text{Str} \left(A_5, G_t^n \right) \right\}. \tag{2.14}
\end{aligned}$$

As we already mentioned above, the secondary form is not unique and it can be modified by the addition of the differential of a one-step-lower-order form $d\beta_{2n+4}$ (1.11). As in the case of ψ_{2n+3} , we will use this freedom in order to simplify our result. Adding $d\beta_{2n+4}$, where

$$\beta_{2n+4} = - \int_0^1 dt \left[n \text{Str} \left(A_{5t}, A, G_t^{n-1} \right) + n(n-1) \text{Str} \left(G_{4t}, A_{3t}, A, G_t^{n-2} \right) \right],$$

we get:

$$\begin{aligned}
\phi_{2n+5} &\sim \phi_{2n+5} + d\beta_{2n+4} = \\
&= \phi_{2n+5} + \int_0^1 dt \left[-n \text{Str} \left(D_t A_{5t}, A, G_t^{n-1} \right) + n \text{Str} \left(A_{5t}, D_t A, G_t^{n-1} \right) - \right. \\
&\quad - n(n-1) \text{Str} \left(A_{5t}, A, D_t G_t, G_t^{n-2} \right) - n(n-1) \text{Str} \left(D_t G_{4t}, A_{3t}, A, G_t^{n-2} \right) - \\
&\quad - n(n-1) \text{Str} \left(G_{4t}, D_t A_{3t}, A, G_t^{n-2} \right) + n(n-1) \text{Str} \left(G_{4t}, A_{3t}, D_t A, G_t^{n-2} \right) - \\
&\quad \left. - n(n-1)(n-2) \text{Str} \left(G_{4t}, A_{3t}, A, D_t G_t, G_t^{n-3} \right) \right],
\end{aligned}$$

where one should use the Bianchi identities $D_t G_t = 0$ and (B.13) of [10]. Next, with the aid of (5.2) and (5.7) we get,

$$\begin{aligned}
\phi_{2n+5} &\sim \phi_{2n+5} + \int_0^1 dt \left[-n \text{Str} \left(G_{6t}, A, G_t^{n-1} \right) + n \text{Str} \left(\{A_{3t}, A_{3t}\}, A, G_t^{n-1} \right) + \right. \\
&\quad + n \text{Str} \left(A_{5t}, \frac{\partial G_t}{\partial t}, G_t^{n-1} \right) - n(n-1) \text{Str} \left([G_t, A_{3t}], A_{3t}, A, G_t^{n-2} \right) - \\
&\quad \left. - n(n-1) \text{Str} \left(G_{4t}^2, A, G_t^{n-2} \right) + n(n-1) \text{Str} \left(G_{4t}, A_{3t}, \frac{\partial G_t}{\partial t}, G_t^{n-2} \right) \right].
\end{aligned}$$

One can see that the first, the third and fifth terms cancel with the first two terms of ϕ_{2n+5} (2.14). With the aid of (B.9) of [10], the second and the forth terms combine to give $Str\left(A_3, \{A, A_{3t}\}, G_t^{n-1}\right)$. Finally using the equation $tD_t A_3 = G_{4t}$ and (5.2) we shall get the following expression for ϕ_{2n+5} :

$$\begin{aligned}
& \int_0^1 dt \left[2n Str\left(G_{4t}, A_3, G_t^{n-1}\right) + Str\left(A_5, G_t^n\right) + n Str\left(\{A, A_{3t}\}, A_{3t}, G_t^{n-1}\right) + \right. \\
& \quad \left. + n Str\left(A_{5t}, \frac{\partial G_t}{\partial t}, G_t^{n-1}\right) + n(n-1) Str\left(G_{4t}, A_{3t}, \frac{\partial G_t}{\partial t}, G_t^{n-2}\right) \right] = \\
& = \int_0^1 dt \left[Str\left(A_5, G_t^n\right) + n Str\left(A_{5t}, \frac{\partial G_t}{\partial t}, G_t^{n-1}\right) + \right. \\
& \quad \left. + n Str\left(D_t A_3 + t\{A, A_3\}, A_{3t}, G_t^{n-1}\right) + n Str\left(G_{4t}, A_3, G_t^{n-1}\right) + \right. \\
& \quad \left. + n(n-1) Str\left(G_{4t}, A_{3t}, \frac{\partial G_t}{\partial t}, G_t^{n-2}\right) \right] = \\
& = \int_0^1 dt \frac{\partial}{\partial t} \left[Str\left(A_{5t}, G_t^n\right) + n Str\left(G_{4t}, A_{3t}, G_t^{n-1}\right) \right].
\end{aligned}$$

Hence, after the integration we get

$$\phi_{2n+5} = Str\left(A_5, G^n\right) + n Str\left(A_3, G_4, G^{n-1}\right). \quad (2.15)$$

By comparing the representations of the secondary form ϕ_{2n+5} in (2.14) and in (2.15) it becomes clear that the last expression is much more simple and transparent. Let us verify that the exterior derivative of the above form leads us back to Ξ_{2n+6} .

$$\begin{aligned}
d\phi_{2n+5} &= Str\left(DA_5, G^n\right) - n Str\left(A_5, DG, G^{n-1}\right) + n Str\left(DG_4, A_3, G^{n-1}\right) + \\
&+ n Str\left(G_4, DA_3, G^{n-1}\right) - n(n-1) Str\left(G_4, A_3, DG, G^{n-2}\right) = \\
&= Str\left(G_6, G^n\right) - Str\left(\{A_3, A_3\}, G^n\right) + n Str\left([G, A_3], A_3, G^{n-1}\right) + \\
&+ n Str\left(G_4^2, G^{n-1}\right) = Str\left(G_6, G^n\right) + n Str\left(G_4^2, G^{n-1}\right) = \Xi_{2n+6},
\end{aligned}$$

where on the second step the second and third terms cancel because of (B.10) of [10].

In order to find out the potential anomalies we have to calculate the gauge variation of the

secondary form ϕ_{2n+5} with respect to the scalar, rank-2 and rank-4 gauge parameters. We have

$$\begin{aligned}
\delta_\xi \phi_{2n+5} &= \delta_\xi \left[\text{Str} \left(A_5, G^n \right) + n \text{Str} \left(G_4, A_3, G^{n-1} \right) \right] = \\
&= \text{Str} \left(\delta_\xi A_5, G^n \right) + n \text{Str} \left(A_5, \delta_\xi G, G^{n-1} \right) + n \text{Str} \left(\delta_\xi G_4, A_3, G^{n-1} \right) + \\
&\quad + n \text{Str} \left(G_4, \delta_\xi A_3, G^{n-1} \right) + n(n-1) \text{Str} \left(G_4, A_3, \delta_\xi G, G^{n-2} \right) = \\
&= \text{Str} \left([A_5, \xi], G^n \right) + n \text{Str} \left(A_5, [G, \xi], G^{n-1} \right) + n \text{Str} \left([G_4, \xi], A_3, G^{n-1} \right) + \\
&\quad + n \text{Str} \left(G_4, [A_3, \xi], G^{n-1} \right) + n(n-1) \text{Str} \left(G_4, A_3, [G, \xi], G^{n-2} \right) = 0.
\end{aligned}$$

Thus, there are no anomalies in the standard gauge symmetry. But there are potential anomalies in the higher-rank gauge symmetries. Indeed,

$$\begin{aligned}
\delta_{\zeta_2} \phi_{2n+5} &= \text{Str} \left(\delta_{\zeta_2} A_5, G^n \right) + n \text{Str} \left(\delta_{\zeta_2} G_4, A_3, G^{n-1} \right) + n \text{Str} \left(G_4, \delta_{\zeta_2} A_3, G^{n-1} \right) = \\
&= 2 \text{Str} \left([A_3, \zeta_2], G^n \right) + n \text{Str} \left([G, \zeta_2], A_3, G^{n-1} \right) + n \text{Str} \left(G_4, D\zeta_2, G^{n-1} \right) = \\
&= \text{Str} \left([A_3, \zeta_2], G^n \right) + n \text{Str} \left(G_4, D\zeta_2, G^{n-1} \right) = \\
&= \text{Str} \left([A_3, \zeta_2], G^n \right) + n d \text{Str} \left(G_4, \zeta_2, G^{n-1} \right) - n \text{Str} \left([G, A_3], \zeta_2, G^{n-1} \right) = \\
&= n d \text{Str} \left(\zeta_2, G_4, G^{n-1} \right)
\end{aligned} \tag{2.16}$$

and

$$\delta_{\zeta_4} \phi_{2n+5} = \text{Str} \left(D\zeta_4, G^n \right) = d \text{Str} \left(\zeta_4, G^n \right). \tag{2.17}$$

Hence the anomalies are:

$$\begin{aligned}
\phi_{2n+4}^{(1)}(\zeta_4, A) &= \text{Str} \left(\zeta_4, G^n \right) \\
\phi_{2n+4}^{(1)}(\zeta_2, A, A_3) &= n \text{Str} \left(\zeta_2, G_4, G^{n-1} \right)
\end{aligned} \tag{2.18}$$

In summary, we have the expressions (2.10) for primary form Ξ_{2n+6} , the expression (2.15) for the secondary form ϕ_{2n+5} and (2.18) for the anomalies.

3 The Form Υ_{2n+8}

In the recent article [21] the authors found the following exact, metric independent forms, linear in G_8 :

$$\begin{aligned}\Upsilon_{10} &= Tr(GG_8 + 3G_4G_6) = Str(G_8, G) + 3Str(G_4, G_6), \\ \Upsilon_{12} &= Tr(G^2G_8 + 3GG_4G_6 + 3GG_6G_4 + 2G_4^3) = \\ &= Str(G_8, G^2) + 6Str(G_4, G_6, G) + 2Str(G_4^3).\end{aligned}\quad (3.1)$$

In order to find the general expression for the forms linear in G_8 let us first find the next form Υ_{14} . For that let us consider the linear combination of all possible rank-14 Lorentz invariant traces which can be constructed in terms of field strength tensors:

$$\begin{aligned}\Upsilon_{14} &= Tr\left(G^3G_8 + c_1GG_6^2 + c_2G_4^2G_6 + c_3G^2G_4G_6 + c_4GG_4GG_6 + c_5G^2G_6G_4 + \right. \\ &\quad \left. + c_6G^4G_6 + c_7GG_4^3 + c_8G^3G_4^2 + c_9GG_4GG_4G + c_{10}G^5G_4 + c_{11}G^7\right).\end{aligned}\quad (3.2)$$

The two terms with the coefficients $c_6 = c_8$ can be dropped since they compose the form Ξ_{14} (2.10). The terms with coefficients c_{10} and c_{11} can also be dropped since they represent the forms Φ_{14} (2.1) and \mathcal{P}_{14} (1.1) respectively. Using the Bianchi identities (5.5) one can calculate the derivatives of the following terms:

$$\begin{aligned}d Tr(G^3G_8) &= 3Tr(G^3(G_6A_3 - A_3G_6 + G_4A_5 - A_5G_4)), \\ d Tr(G^2G_4G_6) &= Tr(G^3(A_3G_6 - G_4A_5) + 2G^2(G_4^2A_3 - G_4A_3G_4) + G^2G_4GA_5 - G^2A_3GG_6), \\ d Tr(GG_4GG_6) &= Tr(GA_3(GG_6G - G^2G_6) + 2GG_4G(G_4A_3 - A_3G_4) + (GG_4G^2 - G^2G_4G)A_5), \\ d Tr(G^2G_6G_4) &= Tr(2G^2(G_4A_3G_4 - A_3G_4^2) + G^2(G_6GA_3 - GG_6A_3) + G^2(GA_5 - A_5G)G_4), \\ d Tr(GG_4^3) &= Tr(G^2(A_3G_4^2 - G_4^2A_3) + GG_4G(A_3G_4 - G_4A_3)),\end{aligned}$$

and see that the following combination is a closed form:

$$d Tr(G^3G_8 + 3G^2G_4G_6 + 3GG_4GG_6 + 3G^2G_6G_4 + 6GG_4^3) = 0. \quad (3.3)$$

Hence,

$$\begin{aligned}\Upsilon_{14} &= Tr(G^3G_8 + 3G^2G_4G_6 + 3G^2G_6G_4 + 3GG_4GG_6 + 6GG_4^3) = \\ &= Str(G_8, G^3) + 9Str(G_4, G_6, G^2) + 6Str(G, G_4^3)\end{aligned}\quad (3.4)$$

and it can be written in terms of symmetric trace. The rest of the terms with the coefficients c_1, c_2, c_9 do not comprise any closed form.

One can check that the Υ_{14} is gauge invariant. In terms of the standard gauge parameter we get:

$$\begin{aligned}
\delta_\xi \Upsilon_{14} = & \text{Tr} \left[[G^3, \xi] G_8 + G^3 \left([G_8, \xi] + 3[G_6, \zeta_2] + 3[G_4, \zeta_4] + [G, \zeta_6] \right) + 3[G^2, \xi] G_4 G_6 + \right. \\
& + 3G^2 \left([G_4, \xi] + [G, \zeta_2] \right) G_6 + 3G^2 G_4 \left([G_6, \xi] + 2[G_4, \zeta_2] + [G, \zeta_4] \right) + \\
& + 3[G^2, \xi] G_6 G_4 + 3G^2 \left([G_6, \xi] + 2[G_4, \zeta_2] + [G, \zeta_4] \right) G_4 + 3G^2 G_6 \left([G_4, \xi] + [G, \zeta_2] \right) + \\
& + 3[G, \xi] G_4 G G_6 + 3G \left([G_4, \xi] + [G, \zeta_2] \right) G G_6 + 3G G_4 [G, \xi] G_6 + \\
& \left. + 3G G_4 G \left([G_6, \xi] + 2[G_4, \zeta_2] + [G, \zeta_4] \right) + 6[G, \xi] G_4^3 + 6G [G_4^3, \xi] \right] = 0. \tag{3.5}
\end{aligned}$$

In an analogous way one can easily prove that Υ_{14} is invariant under the transformations of the higher-tensor gauge parameters $\zeta_2, \zeta_4, \zeta_6$.

Having in hand the series of forms $\Upsilon_{10,12,14}$ one can guess a general expression for Υ_{2n+8} and check that it fulfills all the required properties. We suggest the following general form for Υ_{2n+8} :

$$\Upsilon_{2n+8} = \text{Str} \left(G_8, G^n \right) + 3n \text{Str} \left(G_4, G_6, G^{n-1} \right) + n(n-1) \text{Str} \left(G_4^3, G^{n-2} \right). \tag{3.6}$$

As one can see, each term of the Υ_{2n+8} is separately gauge invariant. Varying over the standard gauge parameter we get:

$$\begin{aligned}
\delta_\xi \text{Str} \left(G_8, G^n \right) &= \text{Str} \left(\delta_\xi G_8, G^n \right) + n \text{Str} \left(G_6, \delta G, G^{n-1} \right) = \\
&= \text{Str} \left([G_8, \xi], G^n \right) + n \text{Str} \left(G_6, [G, \xi], G^{n-1} \right) = 0, \\
\delta_\xi \text{Str} \left(G_4, G_6, G^{n-1} \right) &= \text{Str} \left([G_4, \xi], G_6, G^{n-1} \right) + \text{Str} \left(G_4, [G_6, \xi], G^{n-1} \right) + \\
&\quad + (n-1) \text{Str} \left(G_4, G_6, [G, \xi], G^{n-2} \right) = 0, \\
\delta_\xi \text{Str} \left(G_4^3, G^{n-1} \right) &= 3 \text{Str} \left(\delta_\xi G_4, G_4^2, G^{n-1} \right) + (n-1) \text{Str} \left(G_4^3, \delta_\xi G, G^{n-2} \right) = \\
&= 3 \text{Str} \left([G_4, \xi], G_4^2, G^{n-1} \right) + (n-1) \text{Str} \left(G_4^3, [G, \xi], G^{n-2} \right) = 0 \tag{3.7}
\end{aligned}$$

Analogously, one can check that each term of Υ_{2n+8} is separately invariant under the variation over the higher-tensor gauge parameters. The last two calculations clearly demonstrate the power of the use of the symmetrized traces and of their properties.

Taking the exterior derivative of Υ_{2n+8} one can become convinced that it is a closed form:

$$\begin{aligned}
d\Upsilon_{2n+8} &= Str\left(DG_8, G^n\right) + nStr\left(G_8, DG, G^{n-1}\right) + 3nStr\left(DG_4, G_6, G^{n-1}\right) + \\
&\quad + 3nStr\left(G_4, DG_6, G^{n-1}\right) + 3n(n-1)Str\left(G_4, G_6, DG, G^{n-2}\right) + \\
&\quad + 3n(n-1)Str\left(DG_4, G_4^2, G^{n-2}\right) + n(n-1)(n-2)Str\left(G_4^3, DG, G^{n-3}\right) \\
&= 3Str\left([G_6, A_3], G^n\right) + 3Str\left([G_4, A_5], G^n\right) + Str\left([G, A_7], G^n\right) + \\
&\quad + 3nStr\left([G, A_3], G_6, G^{n-1}\right) + 6nStr\left(G_4, [G_4, A_3], G^{n-1}\right) + \\
&\quad + 3nStr\left(G_4, [G, A_5], G^{n-1}\right) + 3n(n-1)Str\left([G, A_3], G_4, G_4, G^{n-2}\right) = \\
&= 3\left\{Str\left([G_6, A_3], G^n\right) + nStr\left(G_6, [G, A_3], G^{n-1}\right)\right\} + \\
&\quad + 3n\left\{2Str\left([G_4, A_3], G_4, G^{n-1}\right) + (n-1)Str\left(G_4, G_4, [G, A_3], G^{n-2}\right)\right\} + \\
&\quad + 3\left\{Str\left([G_4, A_5], G^n\right) + nStr\left(G_4, [G, A_5], G^{n-1}\right)\right\} + \\
&\quad + \frac{1}{n+1}(n+1)Str\left([G, A_7], G^n\right) = 0. \tag{3.8}
\end{aligned}$$

Again, according to Poincaré's lemma, this equation implies that Υ_{2n+8} can be locally written as an exterior derivative of a certain $(2n+7)$ -form ρ_{2n+7} . In order to find that form we need to differentiate Υ_{2n+8} over the deformation parameter t , as we did in the previous section:

$$\begin{aligned}
\frac{d}{dt}\Upsilon_{2n+8} &= Str\left(\frac{dG_{8t}}{dt}, G_t^n\right) + nStr\left(G_{8t}, \frac{dG_t}{dt}, G_t^{n-1}\right) + 3nStr\left(\frac{dG_{4t}}{dt}, G_{6t}, G_t^{n-1}\right) + \\
&\quad + 3nStr\left(G_{4t}, \frac{dG_{6t}}{dt}, G_t^{n-1}\right) + 3n(n-1)Str\left(G_{4t}, G_{6t}, \frac{dG_t}{dt}, G_t^{n-2}\right) + \\
&\quad + 3n(n-1)Str\left(\frac{dG_{4t}}{dt}, G_{4t}^2, G_t^{n-2}\right) + n(n-1)(n-2)Str\left(G_{4t}, G_{4t}^2, \frac{dG_t}{dt}, G_t^{n-3}\right) \\
&= Str\left(D_t A_7, G_t^n\right) + 6Str\left(\{A_3, A_{5t}\}, G_t^n\right) + Str\left(\{A, A_{7t}\}, G_t^n\right) + \\
&\quad + nStr\left(G_{8t}, D_t A, G_t^{n-1}\right) + 3nStr\left(D_t A_3, G_{6t}, G_t^{n-1}\right) + 3nStr\left(\{A, A_{3t}\}, G_{6t}, G_t^{n-1}\right) + \\
&\quad + 3nStr\left(G_{4t}, D_t A_5, G_t^{n-1}\right) + 3nStr\left(G_{4t}, \{A, A_{5t}\}, G_t^{n-1}\right) + 6nStr\left(G_{4t}, \{A_3, A_{3t}\}, G_t^{n-1}\right) \\
&\quad + 3n(n-1)Str\left(G_{4t}, G_{6t}, D_t A, G_t^{n-2}\right) + 3n(n-1)Str\left(D_t A_3, G_{4t}^2, G_t^{n-2}\right) + \\
&\quad + 3n(n-1)Str\left(\{A, A_{3t}\}, G_{4t}^2, G_t^{n-2}\right) + n(n-1)(n-2)Str\left(G_{4t}^3, D_t A, G_t^{n-3}\right).
\end{aligned}$$

In some of the terms we can extract the covariant exterior derivatives outside the symmetrized traces:

$$\begin{aligned}
\frac{d}{dt}\Upsilon_{2n+8} = & dStr\left(A_7, G_t^n\right) + 6Str\left(\{A_3, A_{5t}\}, G_t^n\right) + Str\left(\{A, A_{7t}\}, G_t^n\right) + \\
& + ndStr\left(G_{8t}, A, G_t^{n-1}\right) - nStr\left(D_t G_{8t}, A, G_t^{n-1}\right) + 3ndStr\left(A_3, G_{6t}, G_t^{n-1}\right) + \\
& + 3nStr\left(A_3, D_t G_{6t}, G_t^{n-1}\right) + 3nStr\left(\{A, A_{3t}\}, G_{6t}, G_t^{n-1}\right) + 3ndStr\left(G_{4t}, A_5, G_t^{n-1}\right) - \\
& - 3nStr\left(D_t G_{4t}, A_5, G_t^{n-1}\right) + 3nStr\left(G_{4t}, \{A, A_{5t}\}, G_t^{n-1}\right) + 6nStr\left(G_{4t}, \{A_3, A_{3t}\}, G_t^{n-1}\right) + \\
& + 3n(n-1)dStr\left(G_{4t}, G_{6t}, A, G_t^{n-2}\right) - 3n(n-1)Str\left(D_t G_{4t}, G_{6t}, A, G_t^{n-2}\right) - \\
& - 3n(n-1)Str\left(G_{4t}, D_t G_{6t}, A, G_t^{n-2}\right) + 3n(n-1)dStr\left(A_3, G_{4t}, G_{4t}, G_t^{n-2}\right) + \\
& + 6n(n-1)Str\left(A_3, D_t G_{4t}, G_{4t}, G_t^{n-2}\right) + 3n(n-1)Str\left(\{A, A_{3t}\}, G_{4t}, G_{4t}, G_t^{n-2}\right) + \\
& + n(n-1)(n-2)dStr\left(G_{4t}^3, A, G_t^{n-3}\right) - 3n(n-1)(n-2)Str\left(D_t G_{4t}, G_{4t}^2, A, G_t^{n-3}\right).
\end{aligned}$$

As one can see, some of the terms are written as exterior derivatives. We shall collect them in the formula below and then combine the rest of the terms in square brackets:

$$\begin{aligned}
\frac{d}{dt}\Upsilon_{2n+8} = & d\left\{ Str\left(A_7, G_t^n\right) + nStr\left(G_{8t}, A, G_t^{n-1}\right) + 3nStr\left(A_3, G_{6t}, G_t^{n-1}\right) + \right. \\
& + 3n\left(G_{4t}, A_5, G_t^{n-1}\right) + 3n(n-1)Str\left(G_{4t}, G_{6t}, A, G_t^{n-2}\right) + \\
& \left. + 3n(n-1)Str\left(A_3, G_{4t}^2, G_t^{n-2}\right) + n(n-1)(n-2)Str\left(G_{4t}^3, A, G_t^{n-3}\right) \right\} + \\
& + 3\left[Str\left(\{A_3, A_{5t}\}, G_t^n\right) + nStr\left(A_3, [G_t, A_{5t}], G_t^{n-1}\right) \right] + \\
& + 3\left[Str\left(\{A_{3t}, A_5\}, G_t^n\right) - nStr\left([G_t, A_{3t}], A_5, G_t^{n-1}\right) \right] + \\
& + Str\left(\{A, A_{7t}\}, G_t^n\right) - nStr\left([G_t, A_{7t}], A, G_t^{n-1}\right) + \\
& + 3n\left[Str\left(\{A, A_{3t}\}, G_{6t}, G_t^{n-1}\right) - Str\left([G_{6t}, A_{3t}], A, G_t^{n-1}\right) - (n-1)Str\left([G_{4t}, A_{3t}], G_{6t}, A, G_t^{n-1}\right) \right] \\
& + 3n\left[Str\left(G_{4t}, \{A, A_{5t}\}, G_t^{n-1}\right) - Str\left([G_{4t}, A_{5t}], A, G_t^{n-1}\right) - (n-1)Str\left(G_{4t}, [G_t, A_{5t}], A, G_t^{n-2}\right) \right] \\
& + 6n\left[Str\left(A_3, [G_{4t}, A_{3t}], G_t^{n-1}\right) + Str\left(G_{4t}, \{A_3, A_{3t}\}, G_t^{n-1}\right) + (n-1)Str\left(A_3, [G_t, A_{3t}], G_{4t}, G_t^{n-2}\right) \right] \\
& + 3n(n-1)\left[Str\left(\{A, A_{3t}\}, G_{4t}, G_{4t}, G_t^{n-2}\right) - 2Str\left(G_{4t}, [G_{4t}, A_{3t}], A, G_t^{n-2}\right) - \right. \\
& \left. - (n-2)Str\left([G_t, A_{3t}], G_{4t}^2, A, G_t^{n-3}\right) \right].
\end{aligned}$$

The terms in the square brackets vanish thanks to the identity (B.10). Therefore we have the

following integral representation for the secondary form:

$$\begin{aligned} \rho_{2n+7} = & \int_0^1 dt \left\{ n \text{Str} \left(G_{8t}, A, G_t^{n-1} \right) + 3n(n-1) \text{Str} \left(G_{4t}, G_{6t}, A, G_t^{n-2} \right) + \right. \\ & + n(n-1)(n-2) \text{Str} \left(G_{4t}^3, A, G_t^{n-3} \right) + 3n \text{Str} \left(G_{6t}, A_3, G_t^{n-1} \right) + \\ & \left. + 3n(n-1) \text{Str} \left(G_{4t}^2, A_3, G_t^{n-2} \right) + 3n \text{Str} \left(G_{4t}, A_5, G_t^{n-1} \right) + \text{Str} \left(A_7, G_t^n \right) \right\}. \end{aligned} \quad (3.9)$$

As we already discussed in the introduction and in the previous section, the secondary forms are defined modulo exact forms and in the given case up to $(2n+7)$ -form $\rho_{2n+7} \sim \rho_{2n+7} + d\gamma_{2n+6}$. Therefore we have to choose an appropriate candidate for γ_{2n+6} . It appears that to simplify the result the exterior derivative of the following form should be subtracted:

$$\begin{aligned} \gamma_{2n+6} = & \int_0^1 dt \left[n \text{Str} \left(A_{7t}, A, G_t^{n-1} \right) + n(n-1)(n-2) \text{Str} \left(G_{4t}^2, A_{3t}, A, G_t^{n-3} \right) + \right. \\ & \left. + n(n-1) \text{Str} \left(G_{6t}, A_{3t}, A, G_t^{n-2} \right) + 2n(n-1) \text{Str} \left(G_{4t}, A_{5t}, A, G_t^{n-2} \right) - n \text{Str} \left(A_{3t}, A_5, G_t^{n-1} \right) \right]. \end{aligned}$$

Subtracting the first two terms of the $d\gamma_{2n+6}$ from ρ_{2n+7} we will get:

$$\begin{aligned} & \text{Str} \left(A_7, G_t^n \right) + n(n-1) \text{Str} \left(G_{4t}^2, A_3, G_t^{n-2} \right) + \\ & + \int_0^1 \left\{ 3n(n-1) \text{Str} \left(G_{4t}, G_{6t}, A, G_t^{n-2} \right) + 3n \text{Str} \left(G_{6t}, A_3, G_t^{n-1} \right) + 3n \text{Str} \left(G_{4t}, A_5, G_t^{n-1} \right) + \right. \\ & \quad + 3n \text{Str} \left(\{A_{3t}, A_{5t}\}, A, G_t^{n-1} \right) - 2n(n-1) \text{Str} \left(\{A, A_{3t}\}, G_{4t}, A_{3t}, G_t^{n-2} \right) - \\ & \quad \left. - 2n(n-1)(n-2) \text{Str} \left([G_t, A_{3t}], G_{4t}, A_{3t}, A, G_t^{n-3} \right) \right\}. \end{aligned}$$

Subtracting now the last three terms of the $d\gamma_{2n+6}$, we will get:

$$\begin{aligned} \rho_{2n+7} = & \text{Str} \left(A_7, G_t^n \right) + n(n-1) \text{Str} \left(G_{4t}, G_{4t}, A_3, G_t^{n-2} \right) + \\ & + n \text{Str} \left(G_{6t}, A_3, G_t^{n-1} \right) + 2n \text{Str} \left(G_{4t}, A_5, G_t^{n-1} \right). \end{aligned} \quad (3.10)$$

Let us check that the exterior derivative of the simplified secondary form gives us back the primary

form Υ_{2n+8} . We have,

$$\begin{aligned}
d\rho_{2n+7} &= Str\left(DA_7, G^n\right) + nStr\left(DG_6, A_3, G^{n-1}\right) + nStr\left(G_6, DA_3, G^{n-1}\right) + \\
&+ 2nStr\left(DG_4, A_5, G^{n-1}\right) + 2nStr\left(G_4, DA_5, G^{n-1}\right) + \\
&+ 2n(n-1)Str\left(DG_4, G_4, A_3, G^{n-2}\right) + n(n-1)Str\left(G_4^2, DA_3, G^{n-2}\right) = \\
&= Str\left(G_8, G^n\right) - (2+1)Str\left(\{A_3, A_5\}, G^n\right) + 2nStr\left([G_4, A_3], A_3, G^{n-1}\right) + \\
&+ nStr\left([G, A_5], A_3, G^{n-1}\right) - nStr\left(G_6, G_4, G^{n-1}\right) + 2nStr\left([G, A_3], A_5, G^{n-1}\right) + \\
&+ 2nStr\left(G_4, G_6, G^{n-1}\right) - 2nStr\left(G_4, \{A_3, A_3\}, G^{n-1}\right) + \\
&+ 2n(n-1)Str\left([G, A_3], G_4, A_3, G^{n-2}\right) + n(n-1)Str\left(G_4^3, G^{n-2}\right) = \Upsilon_{2n+8}.
\end{aligned}$$

Due to (B.10) of [10] the first part of the second term cancels with the sixth one, the second part of the second term cancels with the forth one, and the third term cancels with the ninth one. The secondary form allows to find the potential anomalies of the theory by performing the transgression steps. Thus in order to find out potential anomalies we have to calculate the gauge variation of the secondary form ρ_{2n+7} with respect to the scalar, rank-2, rank-4 and rank-6 gauge parameters:

$$\begin{aligned}
\delta_\xi \rho_{2n+7} &= Str\left([A_7, \xi], G^n\right) + nStr\left(A_7, [G, \xi], G^{n-1}\right) + \\
&+ nStr\left([G_6, \xi], A_3, G^{n-1}\right) + nStr\left(G_6, [A_3, \xi], G^{n-1}\right) + \\
&+ n(n-1)Str\left(G_6, A_3, [G, \xi], G^{n-2}\right) + 2nStr\left([G_4, \xi], A_5, G^{n-1}\right) \\
&+ 2nStr\left(G_4, [A_5, \xi], G^{n-1}\right) + 2n(n-1)Str\left(G_4, A_5, [G, \xi], G^{n-2}\right) + \\
&+ 2n(n-1)Str\left([G_4, \xi], G_4, A_3, G^{n-2}\right) + n(n-1)Str\left(G_4^2, [A_3, \xi], G^{n-2}\right) + \\
&+ n(n-1)(n-2)Str\left(G_4^2, A_3, [G, \xi], G^{n-3}\right) = 0,
\end{aligned} \tag{3.11}$$

where the identity (B.13) of [10] was used. There are no anomalies in the standard gauge symmetry.

The variation over the rank-2 gauge parameter gives:

$$\begin{aligned}
\delta_{\zeta_2} \rho_{2n+7} &= 3\text{Str}\left([A_5, \zeta_2], G^n\right) + 2n\text{Str}\left([G_4, \zeta_2], A_3, G^{n-1}\right) + n\text{Str}\left(G_6, D\zeta_2, G^{n-1}\right) + \\
&\quad + 2n\text{Str}\left([G, \zeta_2], A_5, G^{n-1}\right) + 4n\text{Str}\left(G_4, [A_3, \zeta_2], G^{n-1}\right) + \\
&\quad + 2n(n-1)\text{Str}\left([G, \zeta_2], G_4, A_3, G^{n-2}\right) + n(n-1)\text{Str}\left(G_4^2, D\zeta_2, G^{n-2}\right) = \\
&= \text{Str}\left([A_5, \zeta_2], G^n\right) + n\text{Str}\left(G_6, \zeta_2, G^{n-1}\right) - n\text{Str}\left(DG_6, \zeta_2, G^{n-1}\right) + \\
&\quad + 2n\text{Str}\left(G_4, [A_3, \zeta_2], G^{n-1}\right) + n(n-1)\text{Str}\left(G_4^2, \zeta_2, G^{n-2}\right) - \\
&\quad - 2n(n-1)\text{Str}\left(DG_4, G_4, \zeta_2, G^{n-2}\right) = \\
&= d\left\{n\text{Str}\left(\zeta_2, G_6, G^{n-1}\right) + n(n-1)\text{Str}\left(\zeta_2, G_4^2, G^{n-2}\right)\right\} + \\
&\quad + \text{Str}\left([A_5, \zeta_2], G^n\right) + n\text{Str}\left([A_5, G], \zeta_2, G^{n-1}\right) + \\
&\quad + 2n\text{Str}\left([A_3, G_4], \zeta_2, G^{n-1}\right) + 2n\text{Str}\left(G_4, [A_3, \zeta_2], G^{n-1}\right) - \\
&\quad - 2n(n-1)\text{Str}\left([G, A_3], G_4, \zeta_2, G^{n-2}\right), \tag{3.12}
\end{aligned}$$

where the sum of the third and the fourth terms and the sum of the last three terms vanish due to (B.10). The variation over the rank-4 gauge parameter gives:

$$\begin{aligned}
\delta_{\zeta_4} \rho_{2n+7} &= 3\text{Str}\left([A_3, \zeta_4], G^n\right) + n\text{Str}\left([G, \zeta_4], A_3, G^{n-1}\right) + 2n\text{Str}\left(G_4, D\zeta_4, G^{n-1}\right) = \\
&= 2\text{Str}\left([A_3, \zeta_4], G^n\right) + 2n\text{Str}\left(G_4, \zeta_4, G^{n-1}\right) - 2n\text{Str}\left(DG_4, \zeta_4, G^{n-1}\right) = \\
&= 2\text{Str}\left([A_3, \zeta_4], G^n\right) + 2n\text{Str}\left([A_3, G], \zeta_4, G^{n-1}\right) + 2n\text{Str}\left(\zeta_4, G_4, G^{n-1}\right) \\
&= 2n\text{Str}\left(\zeta_4, G_4, G^{n-1}\right) \tag{3.13}
\end{aligned}$$

and the variation over rank-6 gauge parameter is:

$$\delta_{\zeta_6} \rho_{2n+7} = \text{Str}\left(D\zeta_6, G^n\right) = d\text{Str}\left(\zeta_6, G^n\right). \tag{3.14}$$

Hence the corresponding anomalies are:

$$\begin{aligned}
\rho_{2n+6}^{(1)}(\zeta_6, A) &= \text{Str}\left(\zeta_6, G^n\right), \\
\rho_{2n+6}^{(1)}(\zeta_4, A, A_3) &= 2n\text{Str}\left(\zeta_4, G_4, G^{n-1}\right), \\
\rho_{2n+6}^{(1)}(\zeta_2, A, A_3, A_5) &= n\text{Str}\left(\zeta_2, G_6, G^{n-1}\right) + n(n-1)\text{Str}\left(\zeta_2, G_4^2, G^{n-2}\right) \tag{3.15}
\end{aligned}$$

and there are no anomalies with respect to the standard gauge transformations.

In summary, we have the expressions (3.6) for primary form Υ_{2n+8} , the expression (3.10) for the secondary form ρ_{2n+6} and (3.15) for the anomalies.

4 Conclusion

In this article we are interested in enumerating and classifying *metric independent, gauge invariant and closed forms* in generalized YM theory. The forms that we constructed are defined in various dimensions, are based on non-Abelian tensor gauge fields and are polynomial on the corresponding fields strength tensors - curvature forms. All these forms Φ_{2n+4} , Ξ_{2n+6} and Υ_{2n+8} are analogous to the Pontryagin-Chern-Simons densities \mathcal{P}_{2n} in YM gauge theory (1.1). They are closed forms, but not globally exact.

The secondary characteristic classes ψ_{2n+3} , ϕ_{2n+5} and ρ_{2n+7} have been expressed in integral form (2.4),(2.14) and (3.9) in analogy with the Chern-Simons form (1.4). The secondary forms are not unique, because they can be modified by the addition of the differential of a one-step-lower-order forms. By adding the properly chosen exact forms (2.6), (2.15) and (3.10) respectively to the secondary forms ψ_{2n+3} , ϕ_{2n+5} and ρ_{2n+7} , we are led to much more simple expressions. The gauge variation of the secondary forms can also be found: (2.9), (2.18) and (3.15) yielding the potential anomalies in gauge field theory. The above general considerations should be supplemented by an explicit calculation of loop diagrams involving chiral fermions. The argument in favor of the existence of these potential anomalies is based on the fact that they fulfill Wess-Zumino consistency conditions [4, 6, 7, 8, 9, 18]. The integrals of these forms over the corresponding space-time coordinates provide us with new topological Lagrangians [21] and with a generalization of the Chern-Simons quantum field theory [25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39].

At the same time, these densities constructed on a high-dimensional manifold have their own value. Their integrals represent global geometric invariants suggesting the existence of new topological characterization of the manifolds [6, 7, 10, 16, 17].

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5 Appendix

The gauge transformations of non-Abelian tensor gauge fields were defined in [22, 23, 24]:

$$\begin{aligned}
\delta A &= D\xi, \\
\delta A_3 &= D\zeta_2 + [A_3, \xi] \\
\delta A_5 &= D\zeta_4 + 2[A_3, \zeta_2] + [A_5, \xi], \\
\delta A_7 &= D\zeta_6 + 3[A_3, \zeta_4] + 3[A_5, \zeta_2] + [A_7, \xi], \\
\delta A_9 &= D\zeta_8 + 4[A_3, \zeta_6] + 6[A_5, \zeta_4] + 4[A_7, \zeta_2] + [A_9, \xi], \\
\ldots &\quad \ldots,
\end{aligned} \tag{5.1}$$

where $DA_{2n+1} = dA_{2n+1} + \{A, A_{2n+1}\}$ and the corresponding field-strength tensors are

$$\begin{aligned}
G &= dA + A^2 = DA - A^2 \\
G_4 &= dA_3 + \{A, A_3\} = DA_3, \\
G_6 &= dA_5 + \{A, A_5\} + \{A_3, A_3\} = DA_5 + \{A_3, A_3\}, \\
G_8 &= dA_7 + \{A, A_7\} + 3\{A_3, A_5\} = DA_7 + 3\{A_3, A_5\}, \\
G_{10} &= dA_9 + \{A, A_9\} + 4\{A_3, A_7\} + 3\{A_5, A_5\}, \\
\ldots &\quad \ldots
\end{aligned} \tag{5.2}$$

The general variations of the field-strength tensors are:

$$\begin{aligned}
\delta G &= D(\delta A), \\
\delta G_4 &= D(\delta A_3) + \{A_3, \delta A\}, \\
\delta G_6 &= D(\delta A_5) + \{A_5, \delta A\} + 2\{A_3, \delta A_3\}, \\
\delta G_8 &= D(\delta A_7) + \{A_7, \delta A\} + 3\{A_5, \delta A_3\} + 3\{A_3, \delta A_5\}, \\
\delta G_{10} &= D(\delta A_9) + \{A_9, \delta A\} + 4\{A_7, \delta A_3\} + 6\{A_5, \delta A_5\} + 4\{A_3, \delta A_7\}, \\
\ldots &= \ldots
\end{aligned} \tag{5.3}$$

The gauge transformations of the field-strength tensors follow from (5.3) and (5.1). They are homogeneous:

$$\begin{aligned}
\delta G &= [G, \xi], \\
\delta G_4 &= [G_4, \xi] + [G, \zeta_2], \\
\delta G_6 &= [G_6, \xi] + 2[G_4, \zeta_2] + [G, \zeta_4], \\
\delta G_8 &= [G_8, \xi] + 3[G_6, \zeta_2] + 3[G_4, \zeta_4] + [G, \zeta_6], \\
\delta G_{10} &= [G_{10}, \xi] + 4[G_8, \zeta_2] + 6[G_6, \zeta_4] + 4[G_4, \zeta_6] + [G, \zeta_8], \\
&\dots\dots\dots
\end{aligned} \tag{5.4}$$

The Bianchi identities are given by

$$\begin{aligned}
DG &= 0, \\
DG_4 + [A_3, G] &= 0, \\
DG_6 + 2[A_3, G_4] + [A_5, G] &= 0, \\
DG_8 + 3[A_3, G_6] + 3[A_5, G_4] + [A_7, G] &= 0, \\
DG_{10} + 4[A_3, G_8] + 6[A_5, G_6] + 4[A_7, G_4] + [A_9, G] &= 0, \\
&\dots\dots\dots,
\end{aligned} \tag{5.5}$$

where $DG_{2n} = dG_{2n} + [A, G_{2n}]$. Generalizing Zumino's construction [6], we introduce a one-parameter family of potentials and field-strengths as :

$$\begin{aligned}
A_t &= tA, \quad A_{3t} = tA_3, \quad A_{5t} = tA_5, \quad A_{7t} = tA_7, \quad A_{9t} = tA_9, \\
G_t &= tG + (t^2 - t)A^2, \\
G_{4t} &= tG_4 + (t^2 - t)\{A, A_3\}, \\
G_{6t} &= tG_6 + (t^2 - t)(\{A, A_5\} + \{A_3, A_3\}), \\
G_{8t} &= tG_8 + (t^2 - t)(\{A, A_7\} + 3\{A_3, A_5\}), \\
G_{10t} &= tG_{10} + (t^2 - t)(\{A, A_9\} + 4\{A_3, A_7\} + 3\{A_5, A_5\}), \\
&\dots\dots\dots
\end{aligned} \tag{5.6}$$

The Bianchi identities hold for the deformed fields as well.

$$\begin{aligned}
\frac{\partial G_t}{\partial t} &= dA + 2tA^2 = D_t A, \\
\frac{\partial G_{4t}}{\partial t} &= dA_3 + 2t\{A, A_3\} = D_t A_3 + t\{A, A_3\}, \\
\frac{\partial G_{6t}}{\partial t} &= dA_5 + 2t(\{A, A_5\} + \{A_3, A_3\}) = D_t A_5 + t\{A, A_5\} + 2t\{A_3, A_3\}, \\
\frac{\partial G_{8t}}{\partial t} &= dA_7 + 2t(\{A, A_7\} + 3\{A_3, A_5\}) = D_t A_7 + t\{A, A_7\} + 6t\{A_3, A_5\},
\end{aligned} \tag{5.7}$$

where $D_t A_{2n+1} = dA_{2n+1} + \{A_t, A_{2n+1}\}$. Because we used the properties (B.10) of the symmetrized traces which are defined in [10] we shall present them in the form convenient for our purposes:

$$\sum_{i=1}^n (-1)^{(d_1+\dots+d_{i-1})d_\Theta} \text{Str}(\Lambda_1, \dots, [\Theta, \Lambda_i], \dots, \Lambda_n) = 0, \quad (B.10)$$

where d_i is the rank of the form Λ_i and Θ is an even form. But if both Θ and Λ_i are odd forms, then the commutator should be replaced by the anticommutator. For the exterior derivative we use [10]:

$$d\text{Str}(\Lambda_1, \dots, \Lambda_i, \dots, \Lambda_n) = \sum_{i=1}^n (-1)^{d_1+\dots+d_{i-1}} \text{Str}(\Lambda_1, \dots, D\Lambda_i, \dots, \Lambda_n). \quad (B.13)$$

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